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An introduction to the spinorial chessboard

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Abstract. *The article contains a brief review of the first stage of the authors' research on spinors associated with higher-dimensional geometries and, in particular, on the physical relevance of Cartan's simple (pure) spinors. Historical remarks are followed by a short description of the relation between spinors and null elements. General properties (grading, bilinear forms, charge conjugation) of Clifford algebras associated with real vector spaces with scalar products are described and their double periodicity modulo 8 is exhibited. The latter gives rise to a chessboard arrangement of the algebras; it is shown how the relevant properties of the spin representation of every real Clifford algebra can be simply obtained from those of the representation of an algebra belonging to the chessboard.*

1. INTRODUCTION

Spinors – and structures associated with them – are among the geometrical notions whose importance was recognized as a result of research in physics. For a long time, the interest of physicists in spinors was restricted to three- and four-dimensional spaces (Euclidean and Minkowski). Spinors associated with them have two or four components. Recent work on fundamental interactions and their unification makes essential use of geometries of more than four dimensions. For this reason, spinor structures in higher dimensions and, in particular, Elie Cartan's «simple» or «pure» spinors, have now more chance of

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becoming relevant to physics than they had at the time of the appearance of the article by Brauer and Weyl (1935) and Cartan's (1938) lectures.

This article contains a brief review of the first stage of our research oriented towards physical applications of spinors associated with higher-dimensional geometries. A fuller account is being published under the title *The Spinorial Chessboard* in the Springer-Verlag series of Trieste Notes in Physics. It is intended to be followed by an account of the spinor groups and structures, the geometry of simple spinors and twistors, and of the associated differential equations.

2. A LITTLE OF HISTORY

There is a prehistory of spinors: the period of time, before the discovery of the spin of the electron, when mathematicians considered notions and ideas closely related to those of spin representations (in the present day terminology). It begins probably with Leonhard Euler (1770) and Olinde Rodrigues (1840) who discovered new representations of rotations in three-dimensional space. The latter wrote an equation for a rotation $(x, y, z) \to (x', y', z')$ equivalent to

$$X' = \left[1 + \frac{1}{4}(m^2 + n^2 + p^2)\right]^{-1} U X U^\dagger \tag{1}$$

where

$$U = \begin{pmatrix} 1 + \frac{1}{2}ip & \frac{1}{2}(im + n) \\ \frac{1}{2}(im - n) & 1 - \frac{1}{2}ip \end{pmatrix}, \quad X = \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix}, \tag{2}$$

and similarly for X'. The right hand side of (1) is rational in the components of the vector (m, n, p) parallel to the axis of rotation; the angle of rotation is $\omega = 2 \operatorname{arc\,tg} \frac{1}{2}\sqrt{m^2 + n^2 + p^2}$ and the unitary unimodular matrices $\pm U \cos \frac{1}{2}\omega$ cover the rotation in question. This may be interpreted to mean that Euler and Rodrigues knew that $\mathrm{Spin}(3) = SU(2)$. Formulae for rotations similar to (1) were also known to Carl Ludwig Gauss (cf. Cartan 1908).

The discovery of quaternions by William Rowan Hamilton (1844) led to a much simpler, «spinorial» representation of rotations: if $q = ix + jy + kz$ is a «pure» quaternion and u is a unit quaternion, then

$$q \to u q u^{-1}$$

is a rotation and every rotation can be so obtained. This observation, which can be used to establish the isomorphism $\mathrm{Spin}(3) = Sp(1)$, was made by Arthur

Cayley (1845) who mentioned, however, that the result had been known to Hamilton. Cayley discovered also a quaternionic representation of rotations in four dimensions that was equivalent to the statement $\mathrm{Spin}(4) = Sp(1) \times Sp(1)$ (Cayley 1855). Quaternions are now an important part of the structure of real Clifford algebras. In this context, it is instructive to recall the view of Lord Kelvin (quoted after Kline 1972):

> *«Quaternions came from Hamilton after his really good work had been done; and though beautifully ingenious, have been an unmixed evil to those who have touched them in any way . . . Vector is a useless survival, or offshoot from quaternions, and has never been of the slightest use to any creature».*

The Hamilton-Cayley representation of rotations in 3 and 4 dimensions by quaternions was generalized to higher-dimensional spaces by Rudolf O. Lipschitz (1886) who used for this purpose the associative algebras introduced by William K. Clifford (1878). The algebras considered by Clifford and Lipschitz were generated by n anticommuting «units» e_α with squares equal to -1. In E. Cartan's «Nombres complexes: Exposé, d'après l'article allemand de E. Study (Bonn)» there is a definition and classification of real Clifford algebras of arbitrary signature (Cartan 1908).

The road to spinors initiated by Euler and essentially completed by Clifford and Lipschitz may be described as being based on the idea of *taking the square root of a quadratic form*. Indeed the matrix X given by (1.2) is linear in x, y, z and has the property

$$(3) \qquad X^2 = (x^2 + y^2 + z^2)I$$

where I is the unit 2 by 2 matrix; Clifford algebras provide a universal method of generalizing (3) to higher dimensions and arbitrary signatures.

Spinors have another parentage, related to the study of representations of Lie groups and algebras. The Lie algebras of orthogonal groups have representations which do not lift («integrate») to linear representations of the groups themselves. For example, the Lie algebra of $SO(3)$ is isomorphic to $\mathbb{R}^3$ with the vector product playing the role of the bracket,

$$(4) \qquad [e_1, e_2] = e_3, \text{ etc.}$$

The representation of (4) given by $e_\alpha \to \sigma_\alpha/2i$, where the Pauli matrices are

$$(5) \qquad \sigma = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad i\epsilon = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

does not lift to a representation of $SO(3)$, but integrates to a representation of $SU(2)$, the simply-connected double cover of $SO(3)$, or, in other words, to a two-valued representation of $SO(3)$. Cartan (1913) determined all irreducible representations of the Lie algebras of the groups $SO(n)$ and found that, for every $n > 2$, there are among them representations which do not lift to $SO(n)$. This is so because the groups $SO(n)$ are not simply-connected; the double valuedness comes from

$$\pi_1(SO(n)) = \mathbb{Z}_2 \quad \text{for} \quad n > 2 \tag{6}$$

and Spin(n) is the double cover of $SO(n)$ which is simply-connected for $n > 2$. Cartan's approach was infinitesimal: he considered representations of Lie algebras only. Brauer and Weyl (1935) found global, spinorial represnentations of the groups Spin(n) for all n. This road to spinors may be called *topological*: it is related, in an essential way, to the non-triviality of the fundamental groups π_1 of the groups of rotations. It has the virtue of allowing a generalization of the motion of spinorial representations to general linear groups (Ne'eman 1978). As a manifold, the group $GL^+(n, \mathbb{R})$ of n by n real matrices with positive determinant is homeomorphic to the Cartesian product of manifolds,

$$SO(n) \times \mathbb{R}^{n(n+1)/2}. \tag{7}$$

Therefore, for $n > 2$, $\pi_1(GL^+(n, \mathbb{R})) = \mathbb{Z}_2$ and the group has a simply-connected universal cover $\widetilde{GL}^+(n, \mathbb{R})$ homeomorphic to

$$\text{Spin}(n) \times \mathbb{R}^{n(n+1)/2}. \tag{8}$$

The group $GL^+(n, \mathbb{R})$, for $n > 2$, has no finite-dimensional faithful representations. In other words, spinors associated with the general linear group have an infinity of components. They have the virtue of not requiring, for their definition, any quadratic form or scalar product; they can be contemplated on a «bare» differentiable manifold without metric tensor. The topological approach to spinors is more general than the one based on the idea of linearization of a quadratic form.

The importance of the two-valued representations of the rotation group for physics became clear after the discovery of the intrinsic angular momentum – *spin* – of the electron (Uhlenbeck and Goudsmit 1925) and through the work of Wolfgang Pauli (1927), Paul A.M. Dirac (1928) and many other physicists on wave equations describing the behaviour of fermions, i.e. particles with half-integer spin. According to B.L. van der Waerden (1960), the name *spinor* is due to Paul Ehrenfest.

Hermann Weyl (1929) put forward a relativistic wave equation for massless particles described by a two-component spinor function. Weyl's equation was criticized by Pauli (1933) on the ground that it was not invariant under reflections. Ettore Majorana (1937) introduced another equation, closely related to Weyl's, based on a reality condition equivalent to the identification of the particle and its antiparticle. Two-component equations became accepted in elementary particle physics after the discovery of parity violation in weak interactions.

At first, spinors baffled physicists who, under the influence of relativity theory and despite Lord Kelvin's opinion, were becoming accustomed to scalars, vectors and tensors. In the words of C.G. Darwin (1928):

> *«The relativity theory is based on nothing but the idea of invariance and develops from it the conception of tensors as a matter of necessity; and it is rather disconcerting to find that apparently something has slipped through the net, so that physical quantities exist, which it would be, to say the least, very artificial and inconvenient to express as tensors».*

What is a spinor? Every physicist uses this notion frequently and knows it well, but amazingly diverse definitions of spinors are given in the literature. The differences among the definitions of spinors are more profound than those related to vectors and tensors; for spinors, there are differences in the substance and not only in the form of the definitions.

Geometry and physics require a scheme to deal with *fields* of quantities such as vectors, tensors and spinors. Tensors of various types are first defined in terms of vectors: for example, they may be described as multilinear maps on Cartesian products of vector spaces and their duals. This algebraic definition is then extended to differentiable manifolds by taking the tangent bundle and applying to it the «functor» corresponding to the type of tensors under study. No such functorial or natural construction can be given for spinors because there are topological obstructions to their existence on manifolds. Moreover, the «obvious» algebraic definition of a spinor space may be extended in inequivalent ways to manifolds (Trautman 1987). The algebraic definition may be formulated as follows (Chevalley 1954): assume, for simplicity, that V is a $2m$-dimensional real vector space with a scalar product g_0. The space of (Dirac) spinors of (V, g_0) is the carrier space S_0 of a complex, faithful and irreducible representation of the Clifford algebra $Cl(g_0)$. Since the algebra $Cl(g_0)$ is simple, all such representations are equivalent and the 2^m-dimensional space S_0 is determined up to isomorphism.

There are at least two inequivalent extensions of the algebraic definition of spinors to manifolds. We recall them here for the special case of a $2m$-dimensional oriented manifold M with a positive-definite Riemannian metric tensor g.

(i) The standard definition (Haefliger 1956, Borel and Hirzebruch 1958-60) of a spinor structure on M: it is a *spin prolongation* P of the bundle F_g of orthonormal frames of coherent orientation on M. There are bundle maps

$$\begin{array}{ccccc} & & \mathbb{Z}_2 & & \\ & & \downarrow & & \\ \mathrm{Spin}(2m) & \to & P & \to & M \\ \downarrow & & \downarrow & & \\ SO(2m) & \to & F_g & \to & M \end{array}$$

(see, for example, Dabrowski and Trautman (1986) for details and references). The bundle $\Sigma \to M$ of Dirac spinors is associated with $P \to M$ by the standard representation of Spin$(2m)$ in $S_0 = \mathbb{C}^{2^m}$. The prolongation P exists if, and only if, the second Stiefel-Whitney class of M vanishes.

(ii) If M admits an orthogonal almost complex structure J, then one can define a «Chevalley bundle»

$$S = \Lambda N \subset \Lambda(\mathbb{C} \otimes TM)$$

where N is the totally null subbundle of $\mathbb{C} \otimes TM$ consisting of all complex vectors of the form u-$i\ J(u)$, where $u \in TM$. The bundle $S \to M$ has S_0 as its typical fibre and there is a bundle map

$$Cl(g) \times S \to S$$

making the fibre of $S \to M$ at $x \in M$ into the carrier space of a representation of the Clifford algebra $Cl(g_x)$ associated with (T_xM, g_x), where g_x is the restriction of g to the tangent space T_xM.

The bundles Σ and S are inequivalent: among even-dimensional spheres only those of dimension 2 and 6 admit both Chevalley and Dirac bundles. The Dirac bundles of spheres are all trivial (Gutt 1986), but the Chevalley bundle of S_2 is not. All complex manifolds admit Chevalley bundles defined by their complex structure. In particular, this is true of the even-dimensional complex projective spaces which have no Dirac bundles.

For most purposes, one assumes the standard definition (i). We have mentioned definition (ii) to emphasize a certain non-uniqueness in the notion of spinors on manifolds. The latter definition is closely related to the approach to spinors through differential forms (Ivanenko and Landau 1928, Kähler 1960, Graf 1978) and to the representations of Clifford bundles considered by Karrer (1973).

3. NULL ELEMENTS AND SIMPLE SPINORS

The approach to spinors exposed by Elie Cartan (1938) is based on the use of *null* (1) (light-like, optical) geometrical elements: vectors with vanishing squares

(1) In pure mathematics the adjective «isotropic» is used to denote vectors with vanishing

and linear spaces containing non-zero vectors orthogonal to the space. The connection between spinors and null elements is of fundamental importance for the applications of spinors in the theory of relativity (Penrose 1960, Penrose and Rindler 1984, 1986). It is at the basis of the Newman-Penrose (1962) formalism developed to study and solve Einstein's equations. The discovery of twistors by Pensore (1967) is closely linked to observations concerning a remarkable Robinson congruence of null lines in Minkowski space (Penrose 1987). Twistors have led to deep results, such as new methods for solving both linear and non-linear equations (Penrose and Mac Callum 1972, Ward 1977).

A connection between spinors and null vectors can be illustrated on the old problem of *Pythagorean triples*, i.e. triples x, y, z of positive integers such that

$$x^2 + y^2 = z^2 \tag{9}$$

Equation (9) means that the vector (x, y, z) is null with respect to a scalar product of signature (2,1). It is equivalent to the statement that the symmetric matrix

$$X = \frac{1}{2}\begin{pmatrix} z+y & x \\ x & z-y \end{pmatrix} \tag{10}$$

is of rank 1: $\det X = 0$ and $X \neq 0$. There thus exists a two-component real «spinor» (p, q) such that

$$X = \begin{pmatrix} p \\ q \end{pmatrix}(p\ q) \tag{11}$$

or

$$x = 2pq, \quad y = p^2 - q^2, \quad z = p^2 + q^2. \tag{12}$$

Not only does (12) give a solution of (9), but every Pythagorean triple of relatively prime integers (x, y, z) can be represented as in (12) by choosing a suitable couple of relatively prime integers p and q.

As an example closer to physics, consider the vectors $\mathbf{E}$ and $\mathbf{B}$ of a non-zero electromagnetic field, the complex vector

$$\mathbf{F} = \mathbf{E} + i\mathbf{B} = (F_1, F_2, F_3), \tag{13}$$

and the symmetric matrix

square and also vector spaces consisting of such vectors (Porteous 1981). Physicists refer to such objects as «null». The former choice is somewhat misleading since the word «isotropy» is often used in a different context: there is the isotropy subgroup defined by the action of a group in a space.

$$\Phi = \begin{pmatrix} F_1 + iF_2 & iF_3 \\ iF_3 & F_1 - iF_2 \end{pmatrix} \tag{14}$$

Its determinant,

$$\det \Phi = F_1^2 + F_2^2 + F_3^2$$

vanishes if, and only if, the electromagnetic field is simple or null, i.e. when

$$\mathbf{E} \cdot \mathbf{B} = 0 \quad \text{and} \quad \mathbf{E}^2 = \mathbf{B}^2. \tag{15}$$

If this is so, then there is a complex two-component spinor $\phi = \binom{\phi_1}{\phi_2}$ such that

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} (\phi_1, \phi_2).$$

The spinor $\phi \in \mathbb{C}^2$ is determined by F up to a sign and can be also used to form the Hermitean matrix

$$\psi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} (\bar{\phi}_1, \bar{\phi}_2). \tag{16}$$

Equation (16) can be abbreviated to read $\psi = \phi \phi^\dagger$ and the matrix ψ represented as a linear combination of the three Pauli matrices and the unit matrix $\sigma_0 = I$,

$$\psi = k^\mu \sigma_\mu \quad \text{(summation over } \mu = 0, \ldots, 3) \tag{17}$$

The real vector $k \in \mathbb{R}^4$ with components given by (17) is null with respect to the Minkowski scalar product of signature (1, 3). Moreover,

$$k^0 = |\mathbf{E}| = |\mathbf{B}| \quad \text{and} \quad k^0 \mathbf{k} = \mathbf{E} \times \mathbf{B}, \tag{18}$$

where $(k^1, k^2, k^3) = -\mathbf{k}$. Simple electromagnetic fields characterized by (15) and (18) play a major role in the theory of shear free congruences of null geodesics in Lorentzian manifolds; they give rise to an «optical geometry» and a Cauchy-Riemann structure on the space of null geodesics (Robinson 1961, Penrose 1983a, Trautman 1985, Robinson and Trautman 1986).

To put in perspective these examples, consider the complex vector space $V = \mathbb{C}^{2m}$ with a scalar product g and a faithful irreducible representation

$$\gamma : Cl(2m) \to \mathbb{C}(2^m) \tag{19}$$

of its Clifford algebra $Cl(2m)$. Let $\phi \in S = \mathbb{C}^{2^m}$ be a non-zero Dirac spinor. Its direction dir ϕ defines a vector subspace of V,

$$N(\operatorname{dir} \phi) = \{u \in V \mid \gamma(u)\phi = 0\}. \tag{20}$$

From the basic property of the representation (19),

(21) $\gamma(u)\gamma(v) + \gamma(v)\gamma(u) = 2g(u, v),$

it follows that $N = N(\mathrm{dir}\ \phi)$ is *totally null*, i.e. every vector in N is null. The dimension of N is not larger than m. A necessary condition for N to be of the *maximal dimension* m is that ϕ be a *Weyl spinor*, i.e. an eigenvector of the helicity operator

(22) $\Gamma = i^m \gamma_1 \gamma_2 \ldots \gamma_{2m},$

where $\gamma_\alpha = \gamma(e_\alpha)$ and e_α $(\alpha = 1, \ldots, 2m)$ are the vectors of an orthonormal basis in V embedded in $Cl(2m)$. This condition is also sufficient for $m = 1, 2$, and 3: there is a natural, bijective correspondence between the projective space of Weyl spinors and the set of maxima, totally null planes of the corresponding helicity. For $m \geqslant 4$ the complex dimension $2^{m-1} - 1$ of the projective space of Weyl spinors is larger than the dimension $m(m-1)/2$ of the manifold

(23) $SO(2m)/U(m)$

of maximal totally null planes. Elie Cartan calls a spinor *simple* (in the French edition, Cartan 1938; in the English translation, the adjective *pure* is used) if it defines by (20) a totally null plane of maximal dimension. Cartan shows that a Weyl spinor ϕ is simple if, and only if,

(24) $\langle B\phi, \gamma_{\alpha_1} \gamma_{\alpha_2} \ldots \gamma_{\alpha_p} \phi \rangle = 0$

for all sequences of integers α such that

(25) $1 \leqslant \alpha_1 < \alpha_2 < \ldots < \alpha_p \leqslant 2m$ and $0 \leqslant p \leqslant m - 1.$

Here $B : S \to S^*$ is such that ${}^t\gamma_\alpha = B\gamma_\alpha B^{-1}$ and it is understood that for $p = 0$ condition (24) reduces to

(26) $\langle B\phi, \phi \rangle = 0.$

The m-form with components given by (24) for $p = m$ characterizes the m-dimensional totally null plane associated with the simple spinor ϕ.

In eight dimensions ($m = 4$) equation (26) is the only condition for ϕ to be simple. Here simple spinors lie on a «null cone» in the eight-dimensional space of Weyl spinors; an interesting *triality*, or symmetry between the three eight-dimensional spaces (vector space and two spaces of Weyl spinors), appears in this case (Study 1903, Cartan 1925, Weiss 1933, Chevalley 1954, Tits 1959, Porteous 1981, Penrose and Rindler 1986).

Simple spinors can be defined in a similar manner for real vector spaces with a neutral scalar product. For other signatures, if one insists on staying within the domain of real numbers, the situation is much more complicated and subtle. For example, if the scalar product is positive-definite, then there are no null directions whatsoever and the group $SO(n)$ of rotations acts transitively on the projective space $\mathbb{R}P_{n-1}$ of vector directions. For sufficiently high n, however, the action of Spin(n) on the projective spinor space is not transitive. The «simplicity» of a spinor can be measured by the dimension of its orbit under the action of the spin group: the lower the dimension, the simpler the spinor. Only partial results have been so far obtained on the classification of orbits of Spin(k, l) and the geometrical interpretation of simple spinors in those cases (Porteous 1981, Igusa 1970, Popov 1977, Benn and Tucker 1988, Budinich 1986b, Budinich and Trautman 1986).

4. GENERAL PROPERTIES OF CLIFFORD ALGEBRAS

In this paper, we describe in considerable detail the spinorial representations of the Clifford algebras associated with complex and real vector spaces. We give explicit methods to find the representations for arbitrary dimension and signature. We also present all the essential information about the invariant bilinear and Hermitean forms on the carrier spaces of the representations. Special attention is devoted to the appearance of Weyl and Majorana spinors (of two kinds), to charge conjugation and to the symmetry and signature of the invariant forms. Our main tool is the classical theorem about representations of simple algebras.

To obtain an overall picture of the representations of Clifford algebras it is convenient to divide the study into several steps in such a way that at each step a new structure is introduced.

(i) At first, one forgets about the Clifford algebra everything but its structure of *algebra* $\mathscr{A}$. For any algebra $\mathscr{B}$, we denote by $2\mathscr{B}$ the direct sum $\mathscr{B} \oplus \mathscr{B}$. There are two types of complex algebras,

$$\mathbb{C}(2^m) \quad \text{and} \quad 2\,\mathbb{C}(2^m),$$

and five types of real algebras,

$$\mathbb{R}(2^m),\ 2\mathbb{R}(2^m),\ H(2^m),\ 2H(2^m) \ \text{and}\ \mathbb{C}(2^m).$$

The integer m is simply related to the dimension of the underlying vector space. For example, considered as abstract algebras, the three algebras $Cl(4)$, $Cl_0(4, 1)$ and $Cl_0(2.3)$ are all isomorphic to $\mathbb{C}(4)$. Here and in the sequel $Cl(k, l)$ denotes the real Clifford algebra associated with a scalar product of signature (k, l). Its even subalgebra is denoted by $Cl_0(k, l)$.

(ii) If the Clifford algebra is considered together with its $\mathbb{Z}_2$-grading given by the main automorphism α, then there are still two types of complex algebras, but already eight classes of real algebras, cf. Table I, «The real clock». This provides a classification finer than at the previous step, but one cannot determine the signature of the underlying vector space from the sole knowledge of its graded Clifford algebra $\mathscr{A}_0 \to \mathscr{A}$. For example, the graded algebra

$$2\mathbb{R}(8) \to \mathbb{R}(16)$$

is isomorphic to $Cl_0(8, 0) \to Cl(8, 0)$, $Cl_0(4,4) \to Cl(4,4)$ and $Cl_0(0,8) \to Cl(0,8)$. The class of the real algebra $Cl(k, l)$ depends on

(27) $$k - l \mod 8.$$

(iii) If

(28) $$\gamma : \mathscr{A} \to \operatorname{End} S$$

is a faithful irreducible representation of a simple algebra $\mathscr{A}$ with an involutive antiautomorphism β, then the contragredient representation

$$\check{\gamma} : \mathscr{A} \to \operatorname{End} S^*, \quad \text{where} \quad \check{\gamma}(a) = {}^t\gamma(\beta(a)),$$

is equivalent to γ and there exists an isomorphism $B : S \to S^*$ intertwining γ and $\check{\gamma}$. If $\mathscr{A}$ is central simple, then B is either symmetric or skew; it defines an inner product on S. The symmetry of B depends on the dimension n of the underlying vector space

(29) $${}^tB = \begin{cases} B & \text{for } n = 0, 1, 2, 7 \mod 8 \\ -B & \text{for } n = 3, 4, 5, 6 \mod 8 \end{cases}$$

The double periodicity mod 8 given by (27) and (29) gives rise to a *chessboard* arrangement of real Clifford algebras alluded to in the title of this work and presented in Tables II - V.

(iv) There is a great wealth of structure in a Clifford algebra $\mathscr{A}$ taken together with the vector space V that generates it:

1. The natural linear isomorphisms

(30) $$\mathscr{A} \simeq \Lambda V \simeq \Lambda V^*$$

allow an interpretation of elements of the Clifford algebra as multivectors or forms.

2. The grading, $\mathscr{A} = \mathscr{A}_0 \oplus \mathscr{A}_1$, may be used to define an associated graded or «super» Lie algebra. Its underlying vector space coincides with $\mathscr{A}$ and the graded bracket is

$$[a, b] = ab - (-1)^{pq}\, ba, \quad \text{where} \quad a \in \mathscr{A}_p, \ b \in \mathscr{A}_q,$$

and $p, q = 0$ or 1. Of particular interest is the graded Lie subalgebra

$$\mathcal{L} = K \oplus V \oplus \Lambda^2 V.$$

If $u, v \in V$, then

(31) $$[u, v] = uv + vu = 2g(u, v)$$

so that

$$[K, \mathcal{L}] = 0, \ [V, V] \subset K, \ [V, \Lambda^2 V] \subset V$$

and

$$[\Lambda^2 V, \Lambda^2 V] \subset \Lambda^2 V.$$

The last inclusion means that $\Lambda^2 V$ is an (ungraded) Lie subalgebra: it is the Lie algebra of the orthogonal and spin groups. These groups are also submanifolds of $\mathscr{A}$; we defer their detailed description to subsequent work.

3. If $\mathscr{B}$ is a minimal left ideal of a simple algebra with unity $\mathscr{A}$, then

$$\gamma : \mathscr{A} \to \mathrm{End}\, \mathscr{B}, \quad \text{where } \gamma(a)b = ab,$$

for every $a \in \mathscr{A}$ and $b \in \mathscr{B}$, is a faithful irreducible representation of $\mathscr{A}$. This gives Chevalley's (1954) interpretation of spinors as elements of a minimal (left) ideal of a Clifford algebra.

All Clifford algebras are «supercentral»: numbers (scalars) are the only elements which supercommute with all elements of the Clifford algebra (Wall 1964). If (e_α) is an orthonormal basis for a scalar product of signature (k, l), then the square of the volume element

(32) $$\eta = e_1 e_2 \dots e_{k+l}$$

is

$$\eta^2 = (-1)^{(k-l)(k-l-1)/2}.$$

For $k - l \equiv 2$ or 3 mod 4 the square is negative and η belongs to the centre of $\mathscr{A}_0$ or $\mathscr{A}$, respectively. It may, therefore, be represented by i times the unit endomorphism of the space of Weyl or Dirac spinors.

There are at least two other «independent» ways of introducing complex numbers in quantum theory. The first comes from the observation that energy and momentum are related to translations. Infinitesimal translations are represented by first-order differential operators. To make them (formally) self-adjoint one has to multiply them by i. A related observation is that the Laplacian on compact Riemannian spaces is a negative operator.

Another reason for considering complex wave functions and, in particular, spinor fields, has to do with electromagnetic interactions. According to the gauge, or «minimal interaction» principle, wave equations for charged particles contain the gradient operator d always in the combination $d\text{-}ieA$, where e is the charge and A the potential of the (external) electromagnetic field. The i comes from the fact that the Lie algebra of the group $U(1)$ – the gauge group of electrodynamics – consists of pure imaginary numbers. It is not a trivial or obvious matter that the three i's (spinorial, quantum-mechanical and electromagnetic) are one and the same; but they are as indicated by the successes of the Dirac equation. Similar remarks have recently been made by Chen Ning Yang (1987).

(v) Let $\mathscr{A}$ denote $Cl(k, l)$ or $Cl_0(k, l)$ depending on whether $k + l = 2m$ or $2m + 1$, respectively. The algebra $\mathscr{A}$ is central simple and, therefore, has only one, up to equivalence, irreducible faithful representation. Let (28) be such a representation in a space S of complex dimension 2^m. The complex conjugate representation

$$\bar{\gamma} : \mathscr{A} \to \operatorname{End} \bar{S}$$

is real-equivalent to γ. There thus exists a linear isomorphism $C : S \to \bar{S}$ intertwining γ and $\bar{\gamma}$,

$$\bar{\gamma}(a)C = C\gamma(a), \qquad a \in \mathscr{A}.$$

It is defined up to a complex factor which can be chosen so that

$$\bar{C}C = \begin{cases} I & \text{for } k - l \equiv 0, 1, 2, 7 \mod 8, \\ -I & \text{for } k - l \equiv 3, 4, 5, 6 \mod 8. \end{cases}$$

Depending on whether $\bar{C}C = I$ or $-I$ the representation γ is real or quaternionic. If it is real, then there are Majorana spinors (of the first kind) defined by $C\phi = \pm\bar{\phi}$. For $k - l \equiv 6 \mod 8$ one can define Majorana spinors (of the second kind) as eigenvectors of $C\gamma(\eta)$, where η is the volume element given by (32). There are no Majorana spinors of any kind for $k - l \equiv 3, 4, 5 \mod 8$.

For $k + l = 2m + 1$, the full algebra $Cl(k, l)$ admits an irreducible representation γ in a complex 2^m-dimensional space. This representation is faithful when restricted to the even subalgebra and can be chosen so that

$$\gamma(\eta) = i^{\nu(\nu-1)/2} I,$$

where $\nu \equiv k - l \mod 8$ and $0 \leqslant \nu \leqslant 7$.

Therefore, there is the equivalence of representations,

$$\bar{\gamma} \sim \begin{cases} \gamma & \text{for } \nu = 1 \text{ and } 5, \\ \gamma \circ \alpha & \text{for } \nu = 3 \text{ and } 7, \end{cases}$$

where α is the main automorphism of $Cl(k, l)$.

5. REPRESENTATIONS OF REAL CLIFFORD ALGEBRAS

In this section we give a short summary of the properties of representations of Clifford algebras of real vector spaces in a language familiar to physicists. The 2^m-dimensional spinor space S is identified with $\mathbb{C}^{2^m}$, the endomorphisms γ_α are 2^m by 2^m matrices and the symbols ${}^tA, A^\dagger$ and $\bar{A}$ denote the usual transpose, Hermitean conjugate and complex conjugate of the matrix A, respectively. Therefore $A^\dagger = {}^t\bar{A}$.

If (k, l) is the signature, $k + l = 2m$ or $2m + 1$, then there are $k + l$ Dirac matrices $\gamma_\alpha \in \mathbb{C}(2^m)$ such that

(33a) $$\gamma_\alpha\gamma_\beta + \gamma_\beta\gamma_\alpha = 0 \text{ for } \alpha \neq \beta, \alpha \text{ and } \beta = 1, \ldots, k + l,$$

(33b) $$\gamma_\alpha^2 = I \text{ for } k \text{ values of } \alpha \text{ and } \gamma_\alpha^2 = -I \text{ for } l \text{ values of } \alpha.$$

We do not insist here that the first k values of the label should correspond to Dirac matrices with positive squares; only the total numbers of positive and negative squares matter.

5.1. The case of even-dimensional spaces, $k + l = 2m$

Let $k - l = 8p + \nu$, where p is an integer and $0 \leqslant \nu \leqslant 7$. The matrix

(34) $$\Gamma = i^{\nu(\nu-1)/2}\gamma_1 \ldots \gamma_{2m} \text{ anticommutes with } \gamma_\alpha,$$

and

(35) $$\Gamma^2 = I.$$

There exist invertible matrices $A, B, C, D, E \in \mathbb{C}(2^m)$ such that for every α

(36A) $$\gamma_\alpha^\dagger = A\gamma_\alpha A^{-1},$$

(36B) $${}^t\gamma_\alpha = B\gamma_\alpha B^{-1},$$

(36C) $$\bar{\gamma}_\alpha = C\gamma_\alpha C^{-1},$$

(36D) $$\gamma_\alpha^\dagger = -D\gamma_\alpha D^{-1},$$

(36E) $${}^t\gamma_\alpha = -E\gamma_\alpha E^{-1}.$$

They satisfy

(37B) $${}^tB = (-1)^{m(m-1)/2}B$$

(37E) $${}^tE = (-1)^{m(m+1)/2}E$$

(37Γ) $${}^t\Gamma = (-1)^m B\Gamma B^{-1}$$

The defining properties (36) determine the matrices $A, \ldots, E$ up to complex factors. These factors can be chosen so that

(38) $$\bar{C}C = (-1)^{\nu(\nu-2)/8} I$$

(39A) $$A = \bar{B}C = A^{\dagger}$$

(39B) $$D = \bar{E}C = D^{\dagger}$$

(40) $$E = i^l B\Gamma$$

The remaining freedom is $A \to \lambda A$, $B \to \lambda\mu B$, $C \to \mu C$, $D \to \lambda D$, $E \to \lambda\mu E$, where λ is real $\neq 0$ and μ is complex of unit modulus.

If U is an invertible matrix, $U \in \mathbf{C}(2^m)$, then the matrices

(41) $$'\gamma_\alpha = U^{-1}\gamma_\alpha U$$

have the properties (33). Marking with primes on the left the matrices associated by (36A-E) with the matrices $'\gamma_\alpha$, we have

(42A) $$'A = U^{\dagger} A U,$$

(42B) $$'B = {}^t U B U,$$

(42C) $$'C = \bar{U}^{-1} C U,$$

and similar relations for $'\Gamma$, $'D$ and $'E$.

The Hermitean forms $\varphi^{\dagger} A \varphi$ and $\varphi^{\dagger} D \varphi$, where $\varphi \in \mathbf{C}^{2^m}$, are *neutral* except in the following cases:

(43A) $\varphi^{\dagger} A \varphi$ is *definite* for $l = 0, k > 0$,

(43D) $\varphi^{\dagger} D \varphi$ is *definite* for $k = 0, l > 0$.

These forms restrict to non-degenerate Hermitean forms on the spaces of Weyl spinors, if, and only if, k is even. For odd k, the matrices A and D change the helicity of Weyl spinors.

5.2. The case of odd-dimensional spaces, $k + l = 2m + 1$

Let $k - l = 8p + \nu$, where p is an integer and $1 \leqslant \nu \leqslant 7$. One can choose the matrices $\gamma_1, \ldots, \gamma_{2m+1}$ so that

(44) $$\gamma_1 \cdots \gamma_{2m+1} = i^{\nu(\nu-1)/2} I.$$

There exist matrices A_0, B_0 and C_0 such that, for every α

(45A) $$\gamma_\alpha^{\dagger} = (-1)^l A_0 \gamma_\alpha A_0^{-1}$$

(45B) $${}^t\gamma_\alpha = (-1)^m B_0 \gamma_\alpha B_0^{-1}$$

(45C) $$\bar{\gamma}_\alpha = (-1)^{\nu(\nu-1)/2} C_0 \gamma_\alpha C_0^{-1}$$

and

(46) $$A_0 = \bar{B}_0 C_0 = A_0^\dagger$$

(47) $${}^t B_0 = (-1)^{m(m+1)/2} B_0$$

(48) $$\bar{C}_0 C_0 = (-1)^{(\nu^2 - 1)/8} I$$

The Hermitean form $\varphi^\dagger A_0 \varphi$ is *neutral* except in the case when either $k = 0$ or $l = 0$: it is then *definite*.

5.3. Adding one dimension to an even-dimensional space

Let $k + l = 2m$ and $k - l = 8p + \nu$, as before. The $2m + 1$ matrices

(49) $$\gamma_1, \ldots, \gamma_{2m} \text{ and } \gamma_{2m+1} = \Gamma$$

are Dirac matrices for a space with signature $(k + 1, l)$ and

(50A) $$A_0 = \begin{cases} A & \text{for } l \text{ even,} \\ D & \text{for } l \text{ odd,} \end{cases}$$

(50B) $$B_0 = \begin{cases} B & \text{for } m \text{ even,} \\ E & \text{for } m \text{ odd,} \end{cases}$$

(50C+) $$C_0 = \begin{cases} C & \text{for } \nu = 0 \text{ or } 4, \\ C\Gamma & \text{for } \nu = 2 \text{ or } 6; \end{cases}$$

where the matrices $\Gamma, A, \ldots, E$ are as in §5.1.

Similarly, the $2m + 1$ matrices

$$\gamma_1, \ldots, \gamma_{2m} \text{ and } \gamma_{2m+1} = i\Gamma$$

are Dirac matrices for a space with signature $(k, l + 1)$. The intertwining matrices A_0 and B_0 are as in (50A) and (50B), but

(50C−) $$C_0 = \begin{cases} C & \text{for } \nu = 2 \text{ or } 6, \\ C\Gamma & \text{for } \nu = 0 \text{ or } 4. \end{cases}$$

5.4. Adding a 2-dimensional neutral space

As an example, we give explicitly all relevant quantities for an extension from signature (k, l) to $(k + 1, l + 1)$. We choose an extension of special kind that allows a simultaneous treatment of even- and odd-dimensional spaces. One can take

(51) $$\gamma'_\alpha = \sigma \otimes \gamma_\alpha (\alpha = 1, \ldots, k + l),\ \gamma'_{k+l+1} = \tau \otimes I \text{ and } \gamma'_{k+l+2} = \epsilon \otimes I$$

(i) For $k + l = 2m$ we have

$$\Gamma' = \sigma \otimes \Gamma, \quad C' = I \otimes C$$

(52) $$A' = \tau \otimes D, \quad D' = (-1)^{l} i\epsilon \otimes A$$

$$B' = \tau \otimes E, \quad E' = (-1)^{l+1} i\epsilon \otimes B$$

(ii) For $k + l = 2m + 1$ we have

(53A) $$A_0' = \begin{cases} i\epsilon \otimes A_0 & \text{for } l \text{ even,} \\ \tau \otimes A_0 & \text{for } l \text{ odd,} \end{cases}$$

(53B) $$B_0' = \begin{cases} -i\epsilon \otimes B_0 & \text{for } m \text{ even,} \\ \tau \otimes B_0 & \text{for } m \text{ odd,} \end{cases}$$

(53C) $$C_0' = \begin{cases} I \otimes C_0 & \text{for } \nu = 1 \text{ or } 5, \\ i\sigma \otimes C_0 & \text{for } \nu = 3 \text{ or } 7, \end{cases}$$

where $k - l = 8p + \nu$ and the matrices A_0', B_0' and C_0' are in the same relation to γ_α' as the matrices A_0, B_0 and C_0 are to γ_α, cf. §5.2.

6. THE SPINORIAL CHESSBOARD

There are several «periodicity properties» of real Clifford algebras and their representations. The type of the algebra depends only on $k - l$ mod 8. But the symmetry properties of the invariant bilinear forms depend on $k + l$ mod 8. There is a «double periodicity» in the set of all real Clifford algebras: it is convenient to describe it by referring it to a chessboard.

We define the *spinorial chessboard* to be the set of 64 real algebras

$$\{Cl(k, l) \mid 0 \leqslant k, l \leqslant 7\}$$

where it is understood that $Cl_0(0,0) \to Cl(0,0)$ is the algebra $\mathbb{R} \to \mathbb{R}$, i.e. $Cl_1(0,0) = \{0\}$. In addition to the chessboard – and representations of its elements – we consider the two eight-dimensional Euclidean algebras $Cl(8,0)$ and $Cl(0,8)$. According to the periodicity property, if $k' = k + 8p$ and $l' = l + 8q$, then

(54) $$Cl(k', l') = Cl(k, l) \quad \mathbb{R}(16^{p+q}).$$

Therefore, every Clifford algebra can be represented as in (54), with $Cl(k, l)$ on the chessboard. The significance of this remark goes beyond the mere isomorphism of algebras (54): the representations of $Cl(k', l')$ and the associated bilinear and Hermitean forms can be easily constructed from those of $Cl(k, l)$. Adding eight dimensions makes larger the Clifford algebra and the associated spinor spaces, but preserves their essential properties such as the symmetry of B, type of C, etc.

To make the last statement more precise, consider a vector space $V = \mathbb{R}^8$ with a positive-definite scalar product. The faithful irreducible representation of its Clifford algebra,

$$(55) \qquad Cl(8,0) \to \mathrm{End}\, S,$$

is real so that S can be taken to be a real, 16-dimensional space (of Majorana spinors). Let $(e_1, \ldots, e_8)$ be an orthonormal basis in V. The set of 2^8 products of the form

$$e_{\alpha_1} e_{\alpha_2} \ldots e_{\alpha_p}, \text{ where } 1 \leqslant \alpha_1 < \alpha_2 < \ldots < \alpha_p \leqslant 8,$$

constitute a basis of the algebra. This basis is orthogonal for the scalar product h on $Cl(8,0)$ defined by

$$h(a, b) = Tr\,\gamma(\beta(a)b).$$

Indeed, if

$$a = e_{\alpha_1} \ldots e_{\alpha_p} \text{ and } b = e_{\beta_1} \ldots e_{\beta_q},$$

where

$$1 \leqslant \alpha_1 < \ldots < \alpha_p \leqslant 8 \text{ and } 1 \leqslant \beta_1 < \ldots < \beta_q \leqslant 8,$$

then

$$\beta(a)b = 1 \text{ whenever } p = q \text{ and } \alpha_1 = \beta_1, \ldots, \alpha_p = \beta_p,$$

and

$$Tr\ \gamma(\beta(a)b) = 0 \text{ otherwise.}$$

Therefore, the scalar product h is positive-definite and the symmetric bilinear form B is also positive-definite. We choose a basis in S such that B is represented by a unit matrix with respect to this basis, and we use the basis to identify S with $\mathbb{R}^{16}$ so that the representation (55) can be described as

$$(56) \qquad \theta : Cl(8,0) \to \mathbb{R}(16)$$

and $\check{\theta} = \theta$, i.e. the Dirac matrices

$$\theta_\alpha = \theta(e_\alpha), \quad \alpha = 1, \ldots, 8,$$

are symmetric,

$$^t\theta_\alpha = \theta_\alpha.$$

They may be chosen to be

$$\begin{aligned}
&\theta_1 = \sigma \otimes I \otimes I \otimes I, && \theta_2 = \epsilon \otimes \epsilon \otimes I \otimes I,\\
&\theta_3 = \epsilon \otimes \sigma \otimes \epsilon \otimes I, && \theta_4 = \epsilon \otimes \sigma \otimes \sigma \otimes \sigma,\\
(57) \qquad &\theta_5 = \epsilon \otimes \sigma \otimes \tau \otimes \epsilon, && \theta_6 = \epsilon \otimes \tau \otimes I \otimes \epsilon,\\
&\theta_7 = \epsilon \otimes \tau \otimes \epsilon \otimes \sigma, && \theta_8 = \epsilon \otimes \tau \otimes \epsilon \otimes \tau,
\end{aligned}$$

Their product

$$\Theta = \tau \otimes I \otimes I \otimes I$$

is also symmetric and $\Theta^2 = I$. There is the decomposition

$$\theta_0 = \theta_+ \oplus \theta_-,$$

where

(58) $$\theta_\pm : Cl_0(8,0) \to \mathbb{R}(8)$$

are the inequivalent Weyl representations of the even algebra. Since Θ anticommutes with the Dirac matrices, one can construct a faithful irreducible representation of the opposite algebra

(58*) $$*\theta : Cl(0,8) \to \mathbb{R}(16)$$

by putting

(59) $$*\theta_\alpha = \Theta\theta_\alpha, \quad \alpha = 1, \ldots, 8,$$

so that the Dirac matrices (59) are skew and

(60) $$ {}^{t}{*\theta}_{\alpha} = \Theta *\theta_{\alpha} \Theta^{-1}. $$

Let

(61) $$ \gamma : Cl(k, l) \to \mathrm{End}\, S $$

be a representation of the Clifford algebra $Cl(k, l)$. One can extend it to representations

$$ \gamma' : Cl(k+8, l) \to \mathbb{R}(16) \otimes \mathrm{End}\, S $$

and

$$ \gamma'' : Cl(k, l+8) \to \mathbb{R}(16) \otimes \mathrm{End}\, S $$

by putting

(62a) $$ \gamma'_{\alpha} = \Theta \otimes \gamma_{\alpha} = \gamma''_{\alpha} (\alpha = 1, \ldots, k+l), $$

(62b) $$ \gamma'_{\alpha+k+l} = \theta_{\alpha} \otimes I \quad (\alpha = 1, \ldots, 8), $$

and

(62c) $$ \gamma''_{\alpha+k+l} = \Theta\theta_{\alpha} \otimes I \quad (\alpha = 1, \ldots, 8). $$

Marking with primes or double primes the quantities corresponding to the extensions γ' or γ'', respectively, we obtain for $k+l$ even

$$ \Gamma' = \Theta \otimes \Gamma = \Gamma'' $$

$$ A' = I \otimes A, \quad A'' = \Theta \otimes A $$

$$ B' = I \otimes B, \quad B'' = \Theta \otimes B $$

(63) $$ C' = I \otimes C = C'' $$

$$ D' = \Theta \otimes D, \quad D'' = I \otimes D $$

$$ E' = \Theta \otimes E, \quad E'' = I \otimes E. $$

Adding 8 «positive» or «negative» dimensions preserves the character of A, B, C or D, E, C, respectively. If A or D is definite, then so is A' or D', respectively. There are similar results for $k + l$ odd, namely:

(64A) A'_0 (for l even) and A''_0 (for l odd) $= I \otimes A_0$,

(64B) B'_0 (for m even) and B''_0 (for m odd) $= I \otimes B_0$,

(64C) C'_0 and C''_0 (for $\nu = 1$ or 5) $= I \otimes C_0$,

(65A) A'_0 (for l odd) and A''_0 (for l even) $= \Theta \otimes A_0$,

(65B) B'_0 (for m odd) and B''_0 (for m even) $= \Theta \otimes B_0$,

(65C) C'_0 and C''_0 (for $\nu = 3$ or 7) $= \Theta \otimes C_0$.

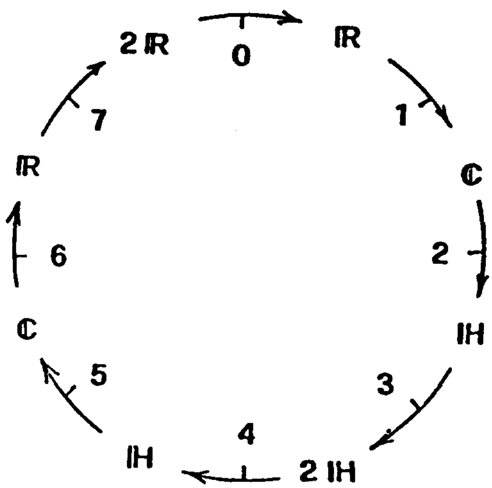

Table I. The real clock (2).

may be used to find the Clifford algebra $Cl(k, l)$ and its even subalgebra $Cl_0(k, l)$: compute first the *hour* μ such that $l - k = 8p + \mu$, where p is an integer and $0 \leqslant \mu \leqslant 7$. The letters adjacent to the hour determine the type of the algebras. The dimension of the full algebra is 2^{k+l}. For example, $Cl_0(3,5) \rightarrow Cl(3,5)$ is $\mathbb{C}(8) \rightarrow H(8)$ because, in this case, $\mu = 2$ and $\dim H(8) = 2^8$.

(2) The complex clock is much simpler: it has a two-hour dial.

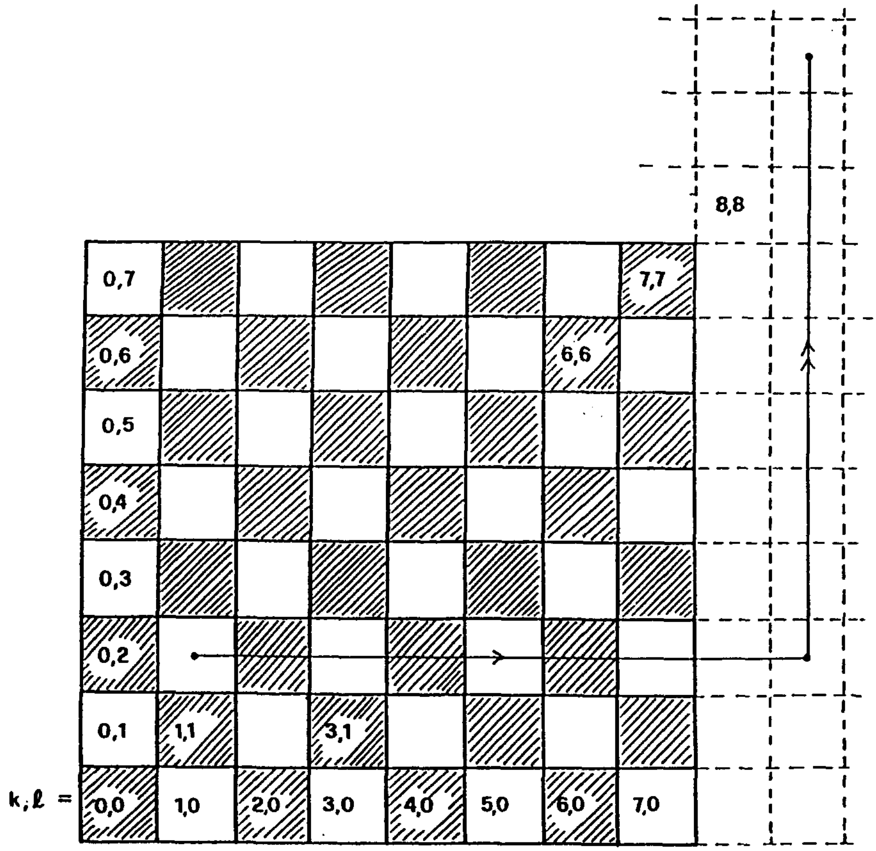

Table II. The Spinorial Chessboard.

Even- and odd-dimensional Clifford algebras $Cl(k, l)$, $0 \leqslant k, l \leqslant 7$, occupy, respectively, black and white squares of the board. For example, the algebra $Cl(3,1)$ of Minkowski space is at the square of the white queen's pawn. Every real Clifford algebra can be reached from one on the board with rook's moves to the right and upwards, each move being by a multiple of eight squares, as described by (62) and (63).

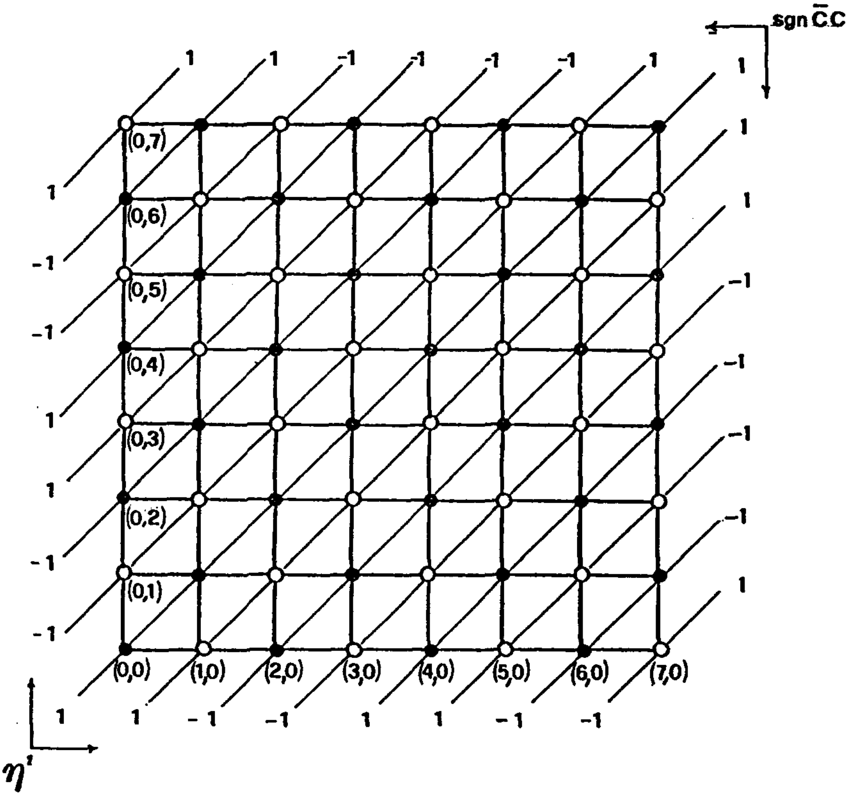

Table III. The structure of the algebras occurring on the chessboard may be determined from the following data.

White and black dots replace here the squares of the chessboard. The figures on the left and lower sides are values of the volume element squared. Those on the right and upper sides determine the type (real if 1, quaternionic if -1) of the full (for $k + l$ even) or even (for $k + l$ odd) Clifford algebra.

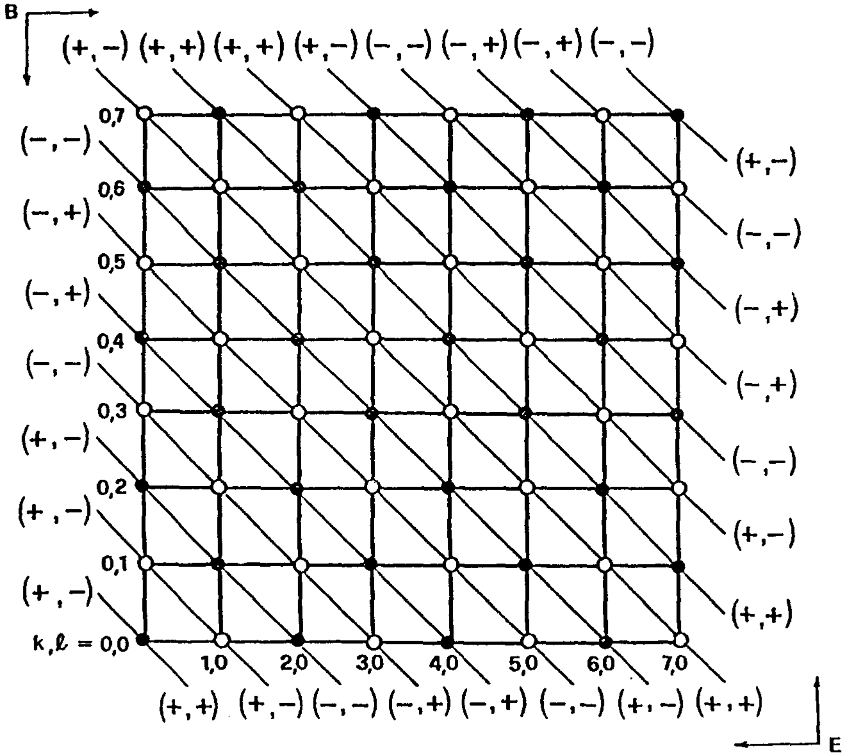

Table IV. The bilinear forms and their symmetries.

The isomorphisms B and E defined by ${}^t\gamma_\alpha = B\gamma_\alpha B^{-1}$ and ${}^t\gamma_\alpha = -E\gamma_\alpha E^{-1}$ are either symmetric or skew and they either commute or anticommute with the helicity operator Γ. These properties are indicated above by pairs (ϵ_1, ϵ_2) where ϵ_1 and $\epsilon_2 = +$ or-. They are defined by ${}^tB = \epsilon_1 B$ and $B\Gamma = \epsilon_2 {}^t\Gamma B$; and similarly for E.

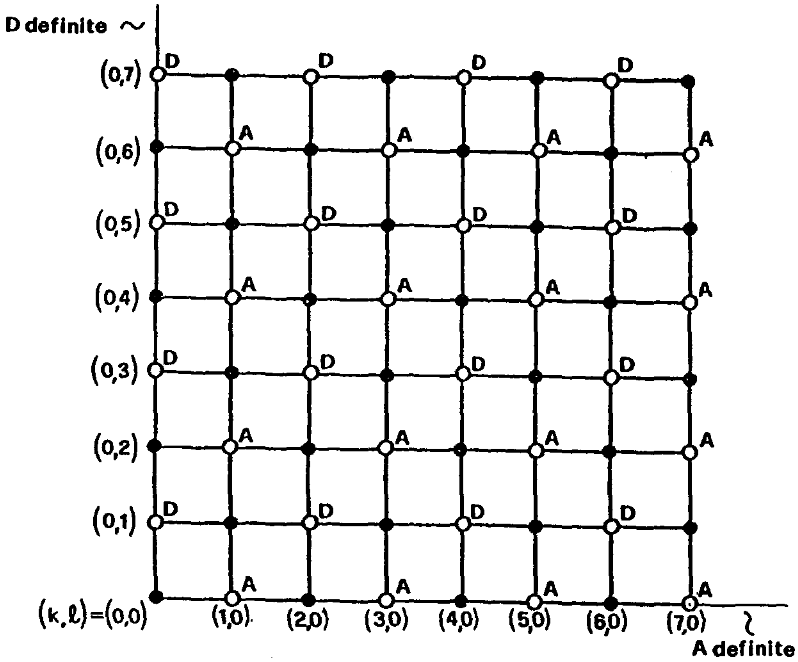

Table V. The Dirac (Hermitean) forms.

The isomorphisms A and D are defined by $\gamma_\alpha^\dagger = A\gamma_\alpha A^{-1}$ and $\gamma_\alpha^\dagger = -D\gamma_\alpha D^{-1}$. They both exist for even dimensional spaces. In an odd number of dimensions, exactly one of the two exists, depending on the parity of k; this is indicated by the letter A or D next to the corresponding white dot. The Hermitean forms $A(\phi, \phi)$ are (positive) definite for the algebras $Cl(k, 0)$; similarly, the Hermitean forms $D(\phi, \phi)$ are (positive) definite for $Cl(0, l)$. Otherwise they are neutral.

7. CONCLUDING REMARKS AND OUTLOOK

Every physicist will agree that spinors are a necessary and important tool in the description of fundamental interactions. The success of the Dirac equation is one of the most beautiful chapters of theoretical physics. Spinors play a major role in essentially all recent attempts at building new models (grand unification, supersymmetry, strings and membranes). They are also very useful in the classical, relativistic theory of gravitation (Penrose and Rindler, 1986). An impressive example of the usefulness of spinor analysis in a new domain has been provided by Edward Witten (1981a) who proved the «positive energy theorem» in Einstein's theory in a manner which is more transparent than the earlier proof due to Schoen and Yau. Thirring (1972) showed that by considering spinors in a five-dimensional space one can obtain *CP* violation in a geometrical way. Recent renewal of interest in generalized Kaluza-Klein theories (cf., for example, the papers by Witten (1981b), Abdus Salam and J. Strathdee (1982), and Steven Weinberg (1983)) has led to considering spinors in spaces of dimension greater than four. In a somewhat different context, one of us (Budinich 1979, 1986b) proposed to consider fields of simple (pure) spinors in suitable higher-dimensional spaces and to relate them to wave-functions of physical particles. There are indications that in this manner a «natural» way of deriving interaction terms of Lagrangians of particles with internal symmetry may be obtained. Attempts have been made to write a differential equation for simple spinors, consistent with the quadratic constraints (24). For example, the method of Lagrange multipliers, applied to a variational principle in 7 space-time dimensions, leads to a Weyl equation for simple spinors with a «mass term» induced by the constraint (26) (cf. Budinich and Trautman 1986 and the references given there). A remark on the possible physical relevance of simple spinors has also been made by A.D. Helfer (1983).

There are some «unexpected» applications of spinors: spinor connections on low-dimensional spheres coincide with simple, topologically non-trivial gauge configurations (Budinich and Trautman 1986). Spinors provide a fine tool for the study of topological properties of manifolds (Atiyah, Bott and Shapiro 1964, Atiyah and Singer 1968). There is a remarkable «spinorial» form of the Enneper-Weierstrass formula for solutions of the equation for minimal surfaces and of its extension to strings (Budinich 1986a, Budinich and Rigoli 1987, and the references given there). It is based on a representation of complex and real null vectors in terms of spinors, analogous to those described in §3.

Considerations such as these convince us that there may be something more to spinors than has been said and seen so far. This view has been put forward, quite a long time ago, by Roger Penrose who pursued the most comprehensive and farthest reaching programme of applying spinors – and their close relatives, twistors – in fundamental physics. We share his view «that we have still not yet seen the full significance of spinors – particularly the 2-component ones – in the basic structure of physical laws» (Penrose 1983b). We are inclined, however, to extend the belief in the significance of spinors to those associated with higher-dimensional geometries and replace the phrase about the 2-component spinors by one referring to simple spinors and the homogeneous spaces mentioned in §3. (Note that, in four-dimensions, simple spinors have two components. More generally, Weyl spinors are simple in neutral spaces of dimension $\leqslant 6$. In particular, twistors are simple).

Our work is an attempt to follow this road. The present article is a preparation for a systematic study of the spin and pin groups and of their representations in relation to simple spinors. We intend to make more precise the idea that the dimension of the orbit is a measure of the simplicity of spinors it contains, use our methods to derive the biquadratic spinor identities (Case 1955), study (simple) spinor fields on homogeneous spaces – such as the one arising from conformal compactification – and consider the possibilities offered by various schemes of dimensional reduction. As many before us, we draw encouragement from the Great Masters. Some of them have already been mentioned. We conclude these remarks with a quotation from Hermann Weyl (1946):

> «*The orthogonal transformations are the automorphisms of Euclidean vector space. Only with the spinors do we strike that level in the theory of its representations on which Euclid himself, flourishing ruler and compass, so deftly moves in the realm of geometric figures*».

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